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Revista OnLine

IPS Instituto Politécnico de Setúbal Escola Superior de Educação

Producing an explanatory proof in a 9^{TH} grade classroom

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Resumo

Este artigo apresenta parte de um estudo fundamentado na problemática da demonstração na matemática escolar. Descreve o modo como quatro alunos do 9.º ano exploraram uma tarefa relacionada com a descoberta de eixos de simetria em várias figuras geométricas. A demonstração, que os mesmos construíram, teve essencialmente uma função explicativa. O papel da professora na negociação do significado de demonstração e da sua necessidade é igualmente analisado. Os alunos desenvolvem primeiro uma compreensão prática sem consciência das razões que fundamentam as afirmações matemáticas e só depois uma compreensão teórica que os conduz à construção de uma demonstração.

Palavras-chave:

Conjecturação, demonstração, prática social, recursos estruturantes, cognição corporizada e matemática

Abstract

This paper presents part of a study that deals with the problem of proof in scholarly mathematics. It describes the way in which four students in 9^{th} grade explored a task related to the discovery of symmetry axes in various geometric figures. The proof constructed by them had essentially an explanatory function. The teacher's role in meaning negotiation of proof and its need is also analysed. One outcome discussed here is that students develop first a practical understanding with no awareness of the reasons underlying mathematical statements and after a theoretical one leading them to a construction of proof.

Key concepts:

Conjecturing, proof, social practice, structuring resources, embodied cognition and mathematics

Theoretical issues

The study's framework is rooted in the theoretical frame of activity theory in the line of Vygotsky and Leont'ev. Drawing on a Vygotskian approach, Wertsch (1991) uses the Bakhtinian construct of 'voice' to emphasise the social origins of individual mental functioning. The process whereby one voice speaks through another voice in a social language is termed 'ventriloquation' by Bakhtin (1981). The word is always half someone else's. So there is a certain interference of one voice on another accompanied by a partial and correlative subordination of the latter.

Mathematics learning is seen as a situated phenomenon (Brown et al., 1988; Lave, 1988; Wenger, 1998). As the school context plays a fundamental role, it is not possible to separate activity, people acting—and respective interactions—and the artefacts that mediate that action. All those dimensions are intrinsically interwoven. The study draws also from embodied cognition perspective (Lakoff et al., 2000) assuming that mathematical concepts are structured by the nature of our bodies and the particular way we function in the world.

Knowledge is not independent of the situation in which it is produced. If situation is structuring cognition, then we can assume also that knowledge and activity are inseparable and mutually constitutive. The centrality of activity in cognition constitutes the base for study theoretical background. It is the mutual interaction between acting and knowing that shapes one another reciprocally (Rodrigues, 1997). Cognition includes the use of representations but it is not based on them. The emphasis falls on the notion of action and the relationship between the subject and the world is redrawn: the subject and the object, that is, the interpreter and the interpreted define one another simultaneously and they are correlatives (Varela, 1988/s.d.); they are not independent nor are separate entities as assumed by the rationalistic perspective.

Proof is inherent to the nature of mathematics as a science (Hanna et al., 1996). The notion of proof has evolved throughout the history of mathematics and it is nowadays the subject of debate among mathematicians. Yet proof maintains a central role in mathematics (Hanna et al., 1996; Thurston, 1995). So the study focuses on the philosophy of mathematics discussing questions such as (a) the nature of mathematical objects; (b) the relationship between the experimental reality, the natural and human world and mathematics; and (c) the issue of truth. The study discusses the epistemological status of proof, assuming mathematics as a human and social construction, but non-arbitral. It is this non-arbitrary that explains the parallelism between the physical reality and the mathematical one (Hersh, 1997). According to Ernest (1993), mathematical knowledge develops through conjectures and refutations (Lakatos, 1994) and relies on linguistic knowledge, conventions and rules.

The study also focuses on the curriculum in general terms and specifically the mathematics curricula, regarding how proof must be integrated. Many mathematics educators attach great importance to proof in the curriculum, claiming that there should be a gradual and continuous transition from justification and explanation activities to the proof itself, from elementary level (Boavida, 2005; Boero et al., 1996; Brocardo, 2001; Harel et al., 2007; Healy et al., 2000; Mariotti, 2000; Veloso, 1998; Yackel et al., 1994). Others (Balacheff, 1991; Duval, 1991), whilst highlighting the prominent role of proof, advocate, however, that the argumentation practice can hinder the learning of proof, assuming the existence of a contradiction between the everyday argumentation and the proof: "mathematical proof should be learned "against" argumentation, bringing students to the awareness of the specificity of mathematical proof" (Balacheff, 1991: p. 189). The more recent curricular documents, in Portugal and in other countries, have attached major importance to proof (DGIDC, 2007; Healy et al., 2000; NCTM, 2000). Two essential reasons justify the relevance of teaching of proof: (a) a more comprehensive vision of the nature of mathematics (de Villiers, 2004; Hanna, 2000; Hanna et al., 1993; 1999; Veloso, 1998), and (b) the promotion of mathematical understanding through the primordial function of proof in mathematics education, the explanatory function (Hanna, 2000; Hanna et al., 1999; Hersh, 1993; 1997; NCTM, 2000).

However, internationally, studies in mathematics education provide empirical evidence that students reveal a great difficulty in understanding the need for proof (Brocardo, 2001), understanding the functions of proof (Harel et al., 2007) and constructing proofs (Healy et al., 2000). The majority of students of various levels (from the more basic to the first years of university level) use particular instances to establish the truth of conjectures they state (Boavida, 2005; Chazan, 1993; Hanna et al. 1993; Harel et al., 2007; Healy et al., 2000; Machado, 2005; Recio et al., 2001; Rodrigues, 1997; 2000). The discussion of mathematical ideas, developed within the small group and orchestrated by the teacher within the class, plays a decisive role (a) in the emergence of proof meaning, (b) in the motivation to proving the mathematical statements (Alibert et al., 1991; Boavida, 2005; Fonseca, 2004), and (c) in changing the spontaneous attitude of students towards construction of proof (Mariotti, 2000). According to Harel et al. (2007: p. 830), "upper elementary school children can deal with proof idea or actions, and . . . high school students can develop meaningful understanding of proof if they are taught appropriately".

1. Aims and Methodology of the Study

The methodology adopted has an interpretative nature because it is adequate for the aims of the study that examine: (1) the role of proof in a classroom in various aspects such as (a) mathematical understanding, (b) validation of mathematical knowledge, and (c) mathematical communication; and (2) the relationship between the construction of proof and the social practice developed in a classroom. The social practice is analysed drawing on a hermeneutic conception of activity and context (Winograd & Flores, 1993) and on a social theory of learning (Wenger, 1998) by questioning (a) the students' group dynamics and (b) the power relations within the students' group. In contexts of work using inquiry pedagogy, the study intends to answer these questions: (1) what is the nature of proof in a school context?; (2) what is the role of proof in students' mathematical activity?; and (3) how does the construction of proof relate to the social practice developed in mathematics classroom?

The analysis unit was proof constructed by students. Through the analysis of scholarly mathematics practice, I tried to understand how students reason in this practice, how the meaning of proof is negotiated, and how the process of proving evolves over time, studying the phenomenon in its natural setting—the mathematics classroom. For that reason, I paid attention to all aspects concerned with students practice: their utterances, their acting, their facial expressions, the mediator resources.

The data were collected in a public school in a class of 9th grade, over a year. Four students were videotaped. The researcher played the role of participant observer, having observed and participated in all mathematical activities of the class during 16 lessons in which inquiry tasks were explored. The data were collected by: (a) video record of mathematical activities of students, (b) audio record of students' dialogues, (c) field notes made by the researcher, (d) video record of students and teacher semi-structured interviews, and (e) documental analysis of the work done by students and of video and audio records.

The video records assume a great importance in data analysis because they allow observations of behaviour procedures (as many as necessary) after they have occurred. They also enable the researcher to capture details that could be ignored by her direct observation in the classroom when the activity was taking place.

2. Discussion of some results

In order to present details data in this paper, I only chose episodes which occurred within the target group related to a single task: the discovery of the number of symmetry axes of various geometric figures, using mirrors.

Structuring resources

Mirrors and the drawing of symmetry axes were structuring resources since they shaped the processes of conjecturing and constructing a proof. According to Lave (1988: pp. 97-8), "such resources are to be found not only in the memory of the person-acting but in activity, in relation with the setting, taking shape at the intersection of multiple realities, produced in conflict and creating value". Students used mirrors, first, to discover the localization of triangle and hexagon symmetry axes. After, they put the mirrors on the correct localization of symmetry axes, without drawing them. They had visualised the symmetry axes: they didn't search for the right localization of the mirror. Here, the act of putting the mirror had merely confirmed what students had visualised before. Since the teacher had demanded that they draw the axes, students drew them using the ruler, dispensing with the mirror. They drew the axes where they saw them through their mental images. The reified character of drawing axes was very important to the conjecture formulation (enabling counting and counting again) and to the production of proof (enabling the observation by where cross the axes). Wenger (1998: p. 58) defines reification as "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'". Reification shapes the experience: drawn axes changed students experience by focusing their attention and enabling higher levels of understanding. According to Wenger (1998: p. 61), the power of reification relies on "its succinctness, its portability, its potential physical persistence, its focusing effect". After drawing the axes, students used the mirror twice as a confirming tool. Sara used it to verify if Ricardo had drawn the pentagon's axes correctly. Ricardo used it to show to Sara that she was wrong when she drew an axe linking two vertices, on the heptagon. First Sara began stammering out a few words pretending to defend its construction using a mathematical argument—"If I put like this it will measure"—but Ricardo didn't let her finish—"Mirror!". He used this artefact as an empirical argument to show the incorrect localization and the correct one. So the mirror validated Ricardo's assertion, arbitrating the discussion between him and Sara.

Conjecturing

The observation of the pattern related to the same number of sides and symmetry axes of the first regular polygons appeared in the table—triangle and square—led pupils to generalise the pattern to the other regular polygons and to state a conjecture: "I can see that the number of sides and the number of axes will be always equal."—said Sara. Three episodes show that students do not regard this conjecture as a suspect proposition: they believe it is true.

Episode 1. This conjecture will be rejected, for a moment, by Sara and Maria, with a lot of resistance, when they were faced with what they judged to be a counter-example. The hexagon was represented in the paper immediately on the right of the square in an unusual position (Figure 1) and they assumed to be the pentagon, following the same order of the table.

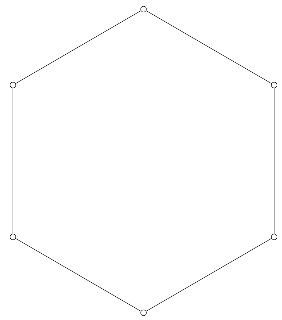

Figure 1. Hexagon position

Sara and Maria reveal difficulty in recognizing the figure in a different position, that is to say, in using the capacity of perceptual constancy (Del Grande, 1990). Sara, perplexed, counted and counted six axes, for a total of five times, where she expected to count just five axes, according to the stated conjecture. Then, disillusioned, Sara wrote the number six, in the table, below the number five respecting to the sides. After a while, Ricardo pointed to the hexagon affirming that it had six sides. And the two girls corrected what they had written before. This episode reveals the conviction on the conjecture. The two pupils had just yielded to the evidence of six axes in a polygon that they thought had five sides after counting the axes a lot of times. In spite of the fact that students believe in the conjecture, they would probably not reject the contradiction by excluding it as a special case, which Lakatos (1994) named 'monster-barring'. However it is impossible to know how their reasoning would evolve in this case because Ricardo's intervention had dissolved the contradiction.

Episode 2. Ricardo had drawn five axes on the heptagon. Bernardo refuted this, after counting them:

B- Silva, this is wrong, Silva. Seven.

R- Seven or five?

B- It's missing to draw yet…

R- Ah! Give it to me! (*he draws two axes more on the heptagon*).

Bernardo did not argue based on referring to the figure symmetry nor did he use the mirror to confirm or to indicate the missing axes. He just saw that his counting did not coincide with the conjecture that affirms that a polygon of seven sides has seven axes. It is a refutation based on the power of the conviction that the conjecture is true.

Episode 3. Sara drew some axes on the octagon. Then, she counted them and continued drawing the other axes until she had a total of eight axes. The conjecture guides her work helping her seeing where they are located, since she assumes that must draw eight axes.

So, when exploring the task, students generalise to *n* sides the pattern observed in concrete polygons, assuming that it is true. They wrote "The conclusions we can achieve that's [sic] the number of sides of regular polygons is always equal to the number of symmetry axes" and in the table they wrote *n* axes below *n* sides. The ongoing process of sequence of regular polygons deals with potential infinity (Fischbein, 2001; Lakoff et al., 2000) because it is an uncompleted sequence since we cannot construct the last polygon corresponding to the final result. However, when students complete the table, they disconnect from the concrete polygons and they think only about the sequence of natural numbers (starting in 3) according to the number of sides and symmetry axes. In this sequence, *n* is conceived by students as a variable assuming any integer value larger than 2, as well as ∞, taken as a number in an enumeration—the actual infinity, conceptualised metaphorically by Basic Metaphor of Infinity (Lakoff et al., 2000) as a final and unique resultant state (the 'natural' and unique ∞ larger than any finite natural number and beyond all of them, taken as an endpoint in an enumeration). Here the Basic Metaphor of Infinity is applied to the special case of enumeration through the addition of the metaphorical completeness. This conjecture starts from the potential infinity of an unending sequence without a final polygon to establish a relationship between the number of sides and the number of symmetry axes of regular polygons. So when students do this generalisation, they conceptualise the ongoing process of the sequence of natural numbers in terms of a completed process, that is to say, they produce the concept of actual infinity.

Also when conjecturing about the infinity of symmetry axes existing in a circle, students apply the Basic Metaphor of Infinity in this special case. We can see a brief extract of their dialogue:

S- Look, the circle... It has axes… It has, it has infinite symmetry axes…

R- The circle?? The circle?? Saaaara!!! Of course it has infinite. The circle is all round.

(…)

B- I don't know if it is infinite.

Why would Bernardo doubt about the infinity of symmetry axes of circle? This is an abstract idea. Even if Bernardo put the mirror on the circle, he would put it a finite number of times. Even if Bernardo draw the circle's symmetry axes, he would draw a finite number of axes. It would be a drawing that represents an abstract idea that can never be put in practice. Considering infinite symmetry axes in a circle implies to consider also infinite points in a circumference. It is more difficult to conceptualise the infinity in a limited object with beginning and ending such as a segment or a circumference than in a straight line. When Ricardo says "The circle is all round", he is conceiving that in a circle it is always possible consider an axis between any two axes and also it is always possible existing a point between any two points of the circumference. The infinity of symmetry axes of a circle is numerable type (Caraça, 1998) characterised by being discrete and infinitely large. The geometric point has no dimensions and consequently it exists an infinity of points between any two points of the circumference. It is a type of infinity characterised by its continuity and density (Caraça, 1998). Given any arc circle, shorter it would be, it is always possible to divide it in half and to get a shorter one. So here we enter the area of infinitely small things. The act of dividing in half is a mental construction that goes on unlimitedly. According to Lakoff et al. (2000), the aspectual system, that characterizes the structure of events as we conceptualise them, is the fundamental source of the concept of infinity. In life, nothing goes on forever. Yet we can conceptualise events as not having completions (imperfective aspect). Let us see the Basic Metaphor of Infinity applied to the arc circle:

Target Domain		*Special Case*
ITERATIVE PROCESSES THAT GO ON AND ON		THE ARC CIRCLE
The beginning state (0)	→	The null circumference ABC_0
State (1) resulting from the initial stage of the process	→	The circumference ABC_1 where the length of arc AB_1 is D_1.
The process: From a prior intermediate state (n-1), produce the next state (n).	→	Form AB_n from AB_{n-1}, by making D_n arbitrarily shorter than D_{n-1}.
The intermediate result after that iteration of the process (the relation between n and $n-1$)	→	$D_n < D_{n-1}$
"The final resultant state" (actual infinity "∞")	→	**D_∞ is infinitely short. The arc AB_∞ is infinitely short.**
Entailment: The final resultant state ("∞") is unique and follows every nonfinal state.	→	**There is a unique arc circle AB_∞ (distance D_∞) that is shorter than any arc circle AB_n (distance D_n) for all finite n.**

The Basic Metaphor of Infinity applied to the arc circle

The act of dividing in half any arc, being an iterative process that goes on indefinitely and produces *n* states, is conceptualised as a complete process with a final resulting state, producing the actual infinity. Therefore the idea of infinite points of the circumference is based on cognitive mechanisms that all people use everyday as the aspectual schemas and the conceptual metaphor (Lakoff et al., 2000). Despite the two different types of infinity in this conjecture—numerable and infinitely large type infinity of axes; continuum and infinitely small type infinity of points of the circumference—we can affirm that it is the dense nature of infinity of points of the circumference that leads to the other one, extending the symmetry axes of circle until the infinity. In spite of the fact that these infinities are different, they are conceptually related and shape each other.

Producing a proof

So students begin their work by conjecturing. In this work phase, they believe their conjecture is true but they do not yet understand why it is true. The experimental work done with mirrors was not enough to foster a deeper mathematical understanding. As Mason et al. (1980) points out, learning only occurs when students reflect on their experimental work. It is this reflective understanding that leads them doing more generalisations and constructing a narrative proof with informal characteristics: “In regular polygons: Odd– the symmetry axes cross vertex-side, and that is the reason why they have the same number of symmetry axes and of vertices; Even- the symmetry axes cross side-side and vertex-vertex and that is the reason why they have half of symmetry axes in relation to the sum of vertices with sides”. For polygons with odd number of sides, students describe by where axes cross them and explain why it is the same number, referring to the vertices, assuming implicitly that in a polygon there are so many vertices as sides. For polygons with even number of sides, they describe by where axes cross them and explain why it is the same number, in spite of the fact that they do not refer explicitly the equality of numbers: this equality would be deduced from the relation reported by them—“half of symmetry axes in relation to the sum of vertices with sides”—and could be expressed algebraically as $\frac{2n}{2} = n$.

This proof has multiple functions (de Villiers, 2001; 2004; Hanna, 2000): verification, explanation and communication. However, for students proof had a unique function: explaining why their conjecture was true. For them, the truth was yet established by conjecturing. It is for that reason that they do not feel need to deduce explicitly the equality of the number of sides and axes for polygons with even number of sides for the purpose of verifying the truth expressed by the conjecture.

The teacher discourse when interacting with group members and the questions posed by the task have a fundamental role in fostering more complex levels of student mathematical thinking. One of the questions of the task (2.c) was of fundamental importance to lead students constructing a proof: “How are the symmetry axes in relation to the vertices and sides? (By where do the axes cross?)”. This question provokes students’ reasoning moving far away from the mere conjecture that the number is the same. I will present here two episodes showing the interactions between the teacher and the target group that illustrate the meaning negotiation of proof.

Episode 4. When students were drawing the axes and completing the table, the teacher said “Besides counting the quantity, how many are, don’t forget to observe how you drew them—you linked what with what, ok?—to be able to answer the following questions”. The teacher focuses students’ attention on the way in which axes cross the polygons. It is evident here the importance of the reified character of drawn axes: the teacher does not demand the observation on the way in which they draw them, at the moment they do the drawing; it is an observation after the drawing. Effectively when students draw the axes they develop a practical understanding; at the moment, they are not aware of the different behaviour of symmetry axes of regular polygons with odd and even number of sides. It is after this practical understanding that students develop a reflective one (Heidegger, 1999) that will enable them to construct a proof for explaining why it is always the same number. However this moment is premature to teacher’s intervention: they continued drawing the axes without observing consciously where they cross. Later, when they faced with the question 2.c), students did not understand its meaning and demanded the teacher’s help—“Teacher, I do not understand the c)”, said Sara. They did not remember the previous teacher’s intervention made too prematurely.

Episode 5. This episode begins with Sara’s request of teacher’s help, referred to in episode 4.

S- Teacher, I do not understand the c).

T- Then, what do you not understand in c)? Go on…

Sara stammers out a few words reading the question and the colleagues read the question 2.c) for the first time..

R- Well, it is here that I said it was the mediatrix.

T- Go on… And are they all? Do they all have that position that we are referring to there? (*points to the chalkboard*) All axes…

R- All axes vertex-side…

T- (…) The question here is: will the symmetry axes’ position be always the same? (…) (*pointing to the triangle)* But, for example, does it link the same elements, always? The same type of elements? Or does it link different elements?

B- Different elements.

T- (*pointing to the triangle again)* That it is a vertex with the opposite side’s midpoint. And the three axes are of same type, isn’t it? (*pointing now to the square with the ruler)* So now here, in the square.

R- (*immediately*) Side with side. Vertex with vertex.

T- And how many, how many do link opposite sides, parallels?

R- Two.

T- Two. And how many do link opposite vertices?

R- Two.

(…)

T- (*pointing to the pentagon*) Now here. Let's go to this of five.

B- It's vertex-side and side-side.

T- Is it always vertex… is it always side-side?

R- No. It's vertex-side.

T- It is always vertex-side, isn't it?

R- Teacher, then the odd is vertex-side and the even vertex-vertex, side-side.

T- Well, go on! You are thinking of a theory, aren't you ? Well, let's see if that theory makes sense. Let's verify it. Let's go! (*she goes away*).

Ricardo looks attentively to the paper where the polygons are represented; he counts and writes something near each polygon.

First, Ricardo was centred on the case of polygons with odd number of sides—"All axes vertex-side…"—and on the fact that symmetry axes cross perpendicularly the midpoint of sides, assuming a symmetry axe as a mediatrix of the side polygon. As we can see, the students' awareness of the different way of axes crossing in polygons with odd and even number of sides emerged only during the dialogue with the teacher when she focused their attention on this aspect, pointing to concrete polygons such as the triangle, square and pentagon. Something appropriated before by the action of drawing the axes has arisen out: the reflective understanding after the practical one, according to Heidegger (1999). And this is why Ricardo has answered so quickly when the teacher pointed to the square. The particular instances are resources that help students in generalising. It was by looking at the concrete polygons pointed out by the teacher that the general principle of axes crossing for all polygons with odd and even number of sides emerged. The teacher has valued Ricardo's generalisation naming it a *theory* but she did not want validate it. She asked students to verify that theory, soliciting them implicitly by testing the generalisation with more specialisations. It was what Ricardo did after the teacher withdrew. There is a continuum movement between specialisation and generalisation. She had also asked about how many axes cross opposite sides and opposite vertices in the square but it was a premature question: they were not yet thinking about the relationship between the Ricardo's theory and the conjecture. The teacher withdrew from the group at the moment she felt students understood the question's meaning and they were in a productive phase of generalisation needed for proof construction. Later, the teacher[1] will negotiate the need of a proof and its meaning: "Now try explaining, based on that, based on crossing in a way to the odd and in another way to the even, why it is always the same number". Students did not feel need for establishing any relationship between Ricardo's theory and the conjecture stated before. And this relation was not asked explicitly in the formulation of the questions of the task. This intervention was fundamental to foster students' reasoning toward a proof.

In closing

Students regard the conjectures as conclusions: they are certain about the truth of conjectures they self say and they do not feel the need to submit them to verification. The teacher has a fundamental role negotiating the need of a proof in Mathematics for guaranteeing the validity of a statement to the generality of cases. It implies an epistemological change: while in other disciplines as Natural Sciences or Physics it seems quite natural to accept a fact as a true one when it is supported by empirical evidence, in mathematics the truth is just accepted on the basis of a theoretical deduction (Hanna, 2000; Hanna et al., 1996; 1999).

Results from Rodrigues (1997) and from the present study are convergent in this point: according to the philosophical perspective of Heidegger (1999), the essence of cognition is "the pre-reflective experience of being *thrown* in a situation of acting" (Winograd & Flores, 1993: p. 97, author italics). The practical understanding is immediate and primary. Students develop a theoretical and reflective understanding only after developing a practical one. The process of generalisation and the process of construction of a proof are intimately associated with that theoretical and reflective understanding.

Students prefer using narrative arguments than algebraic ones. This result is convergent with results of other studies (Healy et al., 2000).

[1] This intervention was made by the researcher playing here a teacher role.

For students, proof had the function of explaining why what they believe is true is indeed true. According to Hanna (2000: p. 8), "in the educational domain, then, it is only natural to view proof first and foremost as explanation, and in consequence to value most highly those proofs which best help to explain".

The group members had different forms of participating in the work. In the target group only one student—Ricardo—had appropriated the proof totally. The other members group used ventriloquation (Wertsch, 1991) incorporating, in part, his discourse given Ricardo's more powerful social status. The team videotaped was analysed in the present study as a community of practice (Wenger, 1998; Wenger et al., 2002): "community is an important element because learning is a matter of belonging as well as an intellectual process, involving the heart as well the head" (Wenger et al.: p. 29). The mathematical communication of proof increased ownership of meaning for all members of the group.

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Anexo

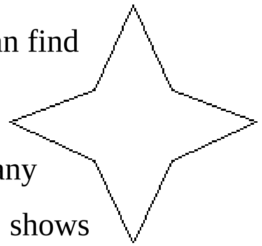
ESCOLA BÁSICA 2-3 C de ÁLVARO VELHO

LET'S INVESTIGATE -MATHEMATICS

Name: ______________________________ **Nº :** __________ **Class;** ____________

The symbols of some makes of car are mathematical figures containing axes of symmetry.

By placing a mirror over an axis of symmetry you can find the whole figure through one part of it.



1. Look at the star on the right and find out how many axes of symmetry it has. Make a drawing which shows what you have discovered.

2. Now use the sheet on the next page which has drawings of regular polygons you already know.

a) Find all the axes of symmetry of each polygon. (Write down your results).

Number of sides of the regular polygon	3	4	5	6	7	8	...	n
Number of axes of symmetry							...	

b) Looking at the table you have filled in, what conclusions can you come to?

c)For each of the regular polygons, explain how the axes of symmetry are placed in relation to the vertices and the sides. (Where do the axes of symmetry pass through?)

3. In the last question you found out how many axes of symmetry there are in an equilateral triangle. Now do experiments with other types of triangles and write down your conclusions about the number of axes of symmetry there are in each of them.

4. There are also many quadrilaterals. For each of them, find out how many axes of symmetry there are. Make a sketch of what you have found out.

5. What about a circle? How many axes of symmetry does it have?

Adapted from APM (2000)(Publisher). *Investigações Matemáticas na sala de aula: Propostas de trabalho.*